Distributional Energy-Momentum Densities of Schwarzschild Space-Time

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Abstract

For Schwarzschild space-time, distributional expressions of energy-momentum densities and of scalar concomitants of the curvature tensors are examined for a class of coordinate systems which includes those of the Schwarzschild and of Kerr-Schild types as special cases. The energy-momentum density $\widetilde{\boldsymbol{T}}_\mu{}^\nu(x)$ of the gravitational source and the gravitational energy-momentum pseudo-tensor density $\widetilde{\boldsymbol{t}}_\mu{}^\nu$ have the expressions $\widetilde{\boldsymbol{T}}_\mu{}^\nu(x) = -Mc^2\delta_\mu{}^0\delta_0{}^\nu\delta^{(3)}(\boldsymbol{x})$ and $\widetilde{\boldsymbol{t}}_\mu{}^\nu = 0$, respectively. In expressions of the curvature squares for this class of coordinate systems, there are terms like $\delta^{(3)}(\boldsymbol{x})/r^3$ and $\left[\delta^{(3)}(\boldsymbol{x})\right]^2$, as well as other terms, which are singular at $\boldsymbol{x} = \boldsymbol{0}$. It is pointed out that the well-known expression $R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\})$ $= 48G^2M^2/c^4r^6$ is not correct, if we define $1/r^6 \stackrel{\text{def}}{=} \lim_{\epsilon\to 0} 1/(r^2+\epsilon^2)^3$.

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1 Introduction

In general relativity, many investigations have been made with regard to exact solutions of the Einstein equation and the singularity structure of space-time, but a distribution theoretical treatment of these has not been developed sufficiently. This is the case even for the well-known Schwarzschild solution, which is given by, in the Schwarzschild coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$,

$$ds^2 = -\left(1 - \frac{a}{\hat{r}}\right)(d\hat{x}^0)^2 + \left(1 - \frac{a}{\hat{r}}\right)^{-1}(d\hat{r})^2 + \hat{r}^2\left[(d\theta)^2 + \sin^2\theta(d\phi)^2\right] . \qquad (1{\cdot}1)$$

Here, a is the Schwarzschild radius $a = 2GM/c^2$ with G, M and c being the Newton gravitational constant, mass of the source, and the light velocity in vacuum Minkowski space-time, respectively. This metric, which is usually called the exterior Schwarzschild metric, is a solution of the Einstein equation in the vacuum region outside a static, spherically symmetric, extended gravitating body, and it is meaningless inside the body.

The metric (1·1) can also describe the gravitational field produced by a point-like particle located at $\hat{\boldsymbol{x}} = \mathbf{0}$. When we say, on the basis of the expression of the curvature square $R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\})$ obtained from the metric (1·1), that $\hat{\boldsymbol{x}} = \mathbf{0}$ is a singularity of the Schwarzschild space-time, the source is considered to be point-like and this metric is regarded as meaningful everywhere in space-time. Then, we have

$$\begin{aligned} \hat{\tilde{\boldsymbol{T}}}_\mu{}^\nu(\hat{x}) &= 0 , \qquad \text{for } \hat{\boldsymbol{x}} \neq \mathbf{0} , \\ \hat{\tilde{\boldsymbol{t}}}_\mu{}^\nu(\hat{x}) &= 0 , \qquad \hat{P}_\mu = (-Mc^2, 0, 0, 0) , \end{aligned} \qquad (1{\cdot}2)$$

where $\hat{\tilde{\boldsymbol{T}}}_\mu{}^\nu$ and $\hat{\tilde{\boldsymbol{t}}}_\mu{}^\nu$ represent the energy-momentum densities of the gravity source and of the gravitational field, respectively, and $\hat{P}_\mu$ is the total energy-momentum of the system. Thus, it is unambiguously expected that the energy-momentum density $\hat{\tilde{\boldsymbol{T}}}_\mu{}^\nu$ of the gravity source has the expression

$$\hat{\tilde{\boldsymbol{T}}}_\mu{}^\nu(\hat{x}) = -Mc^2\delta_\mu{}^0\delta_0{}^\nu\delta^{(3)}(\hat{\boldsymbol{x}}) . \qquad (1{\cdot}3)$$

Nevertheless, this has not been given explicitly in any textbook of general relativity, as far as the present authors know. Also, in the literature, *without giving distribution theoretical examinations*, the expression $R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\}) = 12a^2/r^6$ is given, and the scalars $R^{\mu\nu}(\{\})R_{\mu\nu}(\{\})$ and $R(\{\})$ are not given explicitly, where $R^\rho{}_{\mu\nu\lambda}(\{\})$, $R_{\mu\nu}(\{\})$ and $R(\{\})$ stand for the Riemann-Christoffel curvature tensor, Ricci tensor and scalar curvature, respectively.

Recently, distributional expressions of energy-momentum densities of gravitational sources for the Schwarzschild and Kerr-Newmann space-times have been given by Balasin and Nachbagauer[1, 2, 3, 4] for the Kerr-Schild coordinate system.

The purpose of this paper is to examine, from the distribution theoretical point of view, the energy-momentum densities, and the scalar concomitants $R^{\rho\sigma\mu\nu}(\{\}) \times R_{\rho\sigma\mu\nu}(\{\})$, $R^{\mu\nu}(\{\})R_{\mu\nu}(\{\})$ and $R(\{\})$ of the Schwarzschild space-time by assuming that the metric (1·1) is meaningful everywhere in space-time. The examination is given for the coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$ and for the class of the coordinates (x^0, r, θ, ϕ) obtained by the transformation

$$\hat{x}^0 = \Omega x^0 + f(r) \ , \quad \hat{r} = K(r) \tag{1·4}$$

with Ω being a constant and f and K being functions of the coordinate r. (Note the following: (a)We obtain Kerr-Schild coordinates when $f(r) = a \ln|a - r|, K(r) = r$ and $\Omega = 1$. (b)When $f'(r) = 0, K(r) = r + a/2$ and $\Omega = 1$, we obtain Lanczos coordinates which satisfy the harmonic condition.)

2 Preliminary

We briefly summarize the basics of general relativity, as a preliminary to latter discussion.

In general relativity, the space-time is assumed to be a four-dimensional differentiable manifold endowed with the Lorentzian metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ $(\mu, \nu = 0, 1, 2, 3)$. At each point p of space-time, the metric can be diagonalized as $ds_p^2 = \eta_{\mu\nu}(dX^\mu)_p(dX^\nu)_p$ with $(\eta_{\mu\nu}) \stackrel{\text{def}}{=} \text{diag}\ (-1, 1, 1, 1)$, by choosing the coordinate system $\{X^\mu; \mu = 0, 1, 2, 3\}$ appropriately.

The curvature tensor is given by

$$R^\rho{}_{\sigma\mu\nu}(\{\}) \stackrel{\text{def}}{=} \partial_\mu \left\{ {\rho \atop \sigma\ \nu} \right\} - \partial_\nu \left\{ {\rho \atop \sigma\ \mu} \right\} + \left\{ {\rho \atop \lambda\ \mu} \right\} \left\{ {\lambda \atop \sigma\ \nu} \right\} - \left\{ {\rho \atop \lambda\ \nu} \right\} \left\{ {\lambda \atop \sigma\ \mu} \right\} \tag{2·1}$$

with $\left\{ {\rho \atop \sigma\ \nu} \right\}$ being the Christoffel symbol. The fundamental action integral $\mathbb{I}$ is

$$\mathbb{I} \stackrel{\text{def}}{=} \frac{1}{c} \int (\bar{\mathbb{L}}_G + \mathbb{L}_M) d^4x \ , \tag{2·2}$$

where $\mathbb{L}_M$ is the Lagrangian density of a gravitational source and $\bar{\mathbb{L}}_G$ is the gravitational Lagrangian density given by

$$\bar{\mathbb{L}}_G \stackrel{\text{def}}{=} \frac{1}{2\kappa} \mathbb{G} \ . \tag{2·3}$$

Here κ is the Einstein gravitational constant $\kappa = 8\pi G/c^4$ and $\mathbb{G}$ is defined by

$$\mathbb{G} \stackrel{\text{def}}{=} \sqrt{-g}g^{\mu\nu}\left(\left\{ \begin{matrix} \lambda \\ \mu\ \rho \end{matrix} \right\}\left\{ \begin{matrix} \rho \\ \nu\ \lambda \end{matrix} \right\} - \left\{ \begin{matrix} \lambda \\ \mu\ \nu \end{matrix} \right\}\left\{ \begin{matrix} \rho \\ \lambda\ \rho \end{matrix} \right\}\right) \tag{2·4}$$

with $g \stackrel{\text{def}}{=} \det(g_{\mu\nu})$. There exists the relation

$$\sqrt{-g}R(\{\}) = \mathbb{G} + \partial_\mu \mathbb{D}^\mu \ , \tag{2·5}$$

with

$$\mathbb{D}^\mu \stackrel{\text{def}}{=} -\sqrt{-g}\left(g^{\mu\nu}\left\{ \begin{matrix} \lambda \\ \nu\ \lambda \end{matrix} \right\} - g^{\nu\lambda}\left\{ \begin{matrix} \mu \\ \nu\ \lambda \end{matrix} \right\}\right) \ . \tag{2·6}$$

Also, we have defined the scalar curvature by[1]

$$R(\{\}) \stackrel{\text{def}}{=} R^\mu{}_\mu(\{\}) \tag{2·7}$$

with the Ricci tensor

$$R_{\mu\nu}(\{\}) \stackrel{\text{def}}{=} R^\lambda{}_{\mu\lambda\nu}(\{\}) \ . \tag{2·8}$$

From the action $\mathbb{I}$, the Einstein equation

$$G_\mu{}^\nu(\{\}) \stackrel{\text{def}}{=} R_\mu{}^\nu(\{\}) - \frac{1}{2}\delta_\mu{}^\nu R(\{\}) = \kappa T_\mu{}^\nu \ , \tag{2·9}$$

follows, where $T_\mu{}^\nu$ is defined by

$$T_\mu{}^\nu \stackrel{\text{def}}{=} \frac{\widetilde{\boldsymbol{T}}_\mu{}^\nu}{\sqrt{-g}} \tag{2·10}$$

with

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu \stackrel{\text{def}}{=} 2g_{\mu\lambda}\frac{\delta \mathbb{L}_M}{\delta g_{\lambda\nu}} \tag{2·11}$$

being the energy-momentum density of the gravity source. The energy-momentum pseudo-tensor density[2] $\widetilde{\boldsymbol{t}}_\mu{}^\nu$ of the gravitational field is defined by

$$\widetilde{\boldsymbol{t}}_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu{}^\nu \bar{\mathbb{L}}_G - \frac{\partial \bar{\mathbb{L}}_G}{\partial g_{\sigma\tau,\nu}} g_{\sigma\tau,\mu} \tag{2·12}$$

with $g_{\sigma\tau,\nu} \stackrel{\text{def}}{=} \partial g_{\sigma\tau}/\partial x^\nu$.

[1]Raising and lowering the indices $\mu, \nu, \lambda, \cdots$ is accomplished with the aid of $(g^{\mu\nu}) \stackrel{\text{def}}{=} (g_{\mu\nu})^{-1}$ and $(g_{\mu\nu})$.

[2] One may say, "It is meaningless to deal with $\widetilde{\boldsymbol{t}}_\mu{}^\nu$, because this is not a physically sensible local object." However, the following should be pointed out: (a)For an isolated system, $\widetilde{\boldsymbol{t}}_\mu{}^\nu$ for an asymptotically Minkowskian coordinate system gives the total gravitational energy-momentum when integrated over a space-like surface. (b)Our main concern is not $\widetilde{\boldsymbol{t}}_\mu{}^\nu$, and it plays only an auxiliary role in our discussion (see also the footnote 5 (a) on page 8).

3 Schwarzschild coordinates

Let us first consider the problem in Schwarzschild coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$ by which the Schwarzschild metric is expressed as Eq. (1·1). By using the Cartesian coordinates $(\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)$, which are related to $(\hat{x}^0, \hat{r}, \theta, \phi)$ through the relation

$$\hat{x}^1 = \hat{r}\cos\phi\sin\theta \ , \ \ \hat{x}^2 = \hat{r}\sin\phi\sin\theta \ , \ \ \hat{x}^3 = \hat{r}\cos\theta \ , \tag{3·1}$$

the metric takes the form

$$ds^2 = \hat{g}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu \ , \tag{3·2}$$

where $\hat{g}_{\mu\nu}$ is given by

$$\begin{aligned} \hat{g}_{00} &= -(1-h) \ , \ \ \hat{g}_{0\alpha} = 0 \ , \\ \hat{g}_{\alpha\beta} &= \delta^{\alpha\beta} + h(1-h)^{-1}\frac{\hat{x}^\alpha \hat{x}^\beta}{\hat{r}^2} \ , \quad \alpha, \beta = 1,2,3 \end{aligned} \tag{3·3}$$

with $h \stackrel{\mathrm{def}}{=} a/\hat{r}$.

We know that

$$\begin{aligned} \kappa \hat{\widetilde{\boldsymbol{T}}}_0{}^0 &= -\frac{h'}{\hat{r}} - \frac{h}{\hat{r}^2} \ , \\ \kappa \hat{\widetilde{\boldsymbol{T}}}_0{}^\alpha &= 0 \ , \quad \kappa \hat{\widetilde{\boldsymbol{T}}}_\alpha{}^0 = 0 \ , \\ \kappa \hat{\widetilde{\boldsymbol{T}}}_\alpha{}^\beta &= \delta_\alpha{}^\beta \left(-\frac{h''}{2} - \frac{h'}{\hat{r}} \right) + \frac{\hat{x}^\alpha \hat{x}^\beta}{\hat{r}^2} \left(\frac{h''}{2} - \frac{h}{\hat{r}^2} \right) \ , \end{aligned} \tag{3·4}$$

$$\hat{\mathbb{G}} = 0 \ , \quad \hat{\widetilde{\boldsymbol{t}}}_\mu{}^\nu = 0 \ , \tag{3·5}$$

where the hatted symbols $\hat{\mathbb{G}}, \hat{\widetilde{\boldsymbol{t}}}_\mu{}^\nu$ and $\hat{\widetilde{\boldsymbol{T}}}_\mu{}^\nu$ represent, respectively, $\mathbb{G}, \widetilde{\boldsymbol{t}}_\mu{}^\nu$ and $\widetilde{\boldsymbol{T}}_\mu{}^\nu$ in the coordinate system $\{\hat{x}^\mu; \mu = 0,1,2,3\}$. Also, we have defined $h' \stackrel{\mathrm{def}}{=} dh/d\hat{r}$ and $h'' \stackrel{\mathrm{def}}{=} d^2h/d\hat{r}^2$.

Regularizing the function $h = a/\hat{r}$ as $a/\sqrt{\hat{r}^2 + \epsilon^2}$ with ϵ being a real number, we obtain

$$\begin{aligned} \kappa \hat{\widetilde{\boldsymbol{T}}}_0{}^0(\hat{x};\epsilon) &= -\frac{a\epsilon^2}{\hat{r}^2(\hat{r}^2+\epsilon^2)^{3/2}} \ , \\ \kappa \hat{\widetilde{\boldsymbol{T}}}_0{}^\alpha(\hat{x};\epsilon) &= 0 \ , \quad \kappa \hat{\widetilde{\boldsymbol{T}}}_\alpha{}^0(\hat{x};\epsilon) = 0 \ , \\ \kappa \hat{\widetilde{\boldsymbol{T}}}_\alpha{}^\beta(\hat{x};\epsilon) &= \delta_\alpha{}^\beta \frac{3a\epsilon^2}{2(\hat{r}^2+\epsilon^2)^{5/2}} - \frac{\hat{x}^\alpha \hat{x}^\beta}{\hat{r}^2} \frac{a\epsilon^2}{(\hat{r}^2+\epsilon^2)^{5/2}} \left(\frac{5}{2} + \frac{\epsilon^2}{\hat{r}^2} \right) \ , \end{aligned} \tag{3·6}$$

where $\hat{\widetilde{\boldsymbol{T}}}_\mu{}^\nu(\hat{x};\epsilon)$ stands for the regularized $\hat{\widetilde{\boldsymbol{T}}}_\mu{}^\nu$. Equations (3·4) and (3·5) have been obtained by the use of Eqs. (2·4), (2·12) and (2·9) with Eq. (2·10).

From Eq. (3·6), the desired expression

$$\hat{\tilde{\boldsymbol{T}}}_{\mu}{}^{\nu}(\hat{x}) \stackrel{\rm def}{=} \lim_{\epsilon\to 0} \hat{\tilde{\boldsymbol{T}}}_{\mu}{}^{\nu}(\hat{x};\epsilon) = -Mc^2\delta_{\mu}{}^{0}\delta_{0}{}^{\nu}\delta^{(3)}(\hat{\boldsymbol{x}}) \qquad (1\cdot 3)$$

follows, where use has been made of the relations[3]

$$\begin{aligned} \lim_{\epsilon\to 0}\left[\frac{\epsilon^2}{\hat{r}^2(\hat{r}^2+\epsilon^2)^{3/2}}\right] &= 4\pi\delta^{(3)}(\hat{\boldsymbol{x}}) \ , \\ \lim_{\epsilon\to 0}\left[\frac{\epsilon^2}{(\hat{r}^2+\epsilon^2)^{5/2}}\right] &= \frac{4}{3}\pi\delta^{(3)}(\hat{\boldsymbol{x}}) \ , \end{aligned} \qquad (3\cdot 7)$$

$$\lim_{\epsilon\to 0}\left\{\frac{\hat{x}^\alpha\hat{x}^\beta}{\hat{r}^2}\left[\frac{3}{2}\frac{\epsilon^2}{(\hat{r}^2+\epsilon^2)^{5/2}} + \frac{\epsilon^2}{\hat{r}^2(\hat{r}^2+\epsilon^2)^{3/2}}\right]\right\} = 2\pi\delta^{\alpha\beta}\delta^{(3)}(\hat{\boldsymbol{x}}) \ . \qquad (3\cdot 8)$$

The regularized scalar quantities $\hat{R}(\{\};\epsilon)$, $\hat{R}^{\mu\nu}(\{\};\epsilon)\hat{R}_{\mu\nu}(\{\};\epsilon)$ and $\hat{R}^{\rho\sigma\mu\nu}(\{\};\epsilon)\hat{R}_{\rho\sigma\mu\nu}(\{\};\epsilon)$ are calculated as

$$\begin{aligned} \hat{R}(\{\};\epsilon) &= -\frac{3a\epsilon^2}{(\hat{r}^2+\epsilon^2)^{5/2}} + \frac{2a\epsilon^2}{\hat{r}^2(\hat{r}^2+\epsilon^2)^{3/2}} \ , \\ \hat{R}^{\mu\nu}(\{\};\epsilon)\hat{R}_{\mu\nu}(\{\};\epsilon) &= \frac{1}{2}\left[\frac{3a\epsilon^2}{(\hat{r}^2+\epsilon^2)^{5/2}}\right]^2 + 2\left[\frac{a\epsilon^2}{\hat{r}^2(\hat{r}^2+\epsilon^2)^{3/2}}\right]^2 \ , \\ \hat{R}^{\rho\sigma\mu\nu}(\{\};\epsilon)\hat{R}_{\rho\sigma\mu\nu}(\{\};\epsilon) &= \frac{4a^2}{\hat{r}^2+\epsilon^2}\left[\frac{2}{(\hat{r}^2+\epsilon^2)^2} + \frac{1}{\hat{r}^4}\right] - \frac{12a^2\epsilon^2}{(\hat{r}^2+\epsilon^2)^4} \\ &\quad + \frac{9a^2\epsilon^4}{(\hat{r}^2+\epsilon^2)^5} \ , \end{aligned} \qquad (3\cdot 9)$$

respectively. Thus, the scalar curvature has the well-defined limit

$$\hat{R}(\{\}) \stackrel{\rm def}{=} \lim_{\epsilon\to 0}\hat{R}(\{\};\epsilon) = 4\pi a\delta^{(3)}(\hat{\boldsymbol{x}}) \ . \qquad (3\cdot 10)$$

However, the quadratic scalars do not have well-defined limits, which can be symbolically written as

$$\begin{aligned} \hat{R}^{\mu\nu}(\{\})\hat{R}_{\mu\nu}(\{\}) &\stackrel{\rm def}{=} \lim_{\epsilon\to 0}\hat{R}^{\mu\nu}(\{\};\epsilon)\hat{R}_{\mu\nu}(\{\};\epsilon) \sim 40\pi^2a^2\left[\delta^{(3)}(\hat{\boldsymbol{x}})\right]^2 \ , \\ \hat{R}^{\rho\sigma\mu\nu}(\{\})\hat{R}_{\rho\sigma\mu\nu}(\{\}) &\stackrel{\rm def}{=} \lim_{\epsilon\to 0}\hat{R}^{\rho\sigma\mu\nu}(\{\};\epsilon)\hat{R}_{\rho\sigma\mu\nu}(\{\};\epsilon) \\ &\sim \frac{12a^2}{\hat{r}^6} + \frac{16\pi a^2}{3}\frac{1}{\hat{r}^3}\delta^{(3)}(\hat{\boldsymbol{x}}) + 16\pi^2a^2\left[\delta^{(3)}(\hat{\boldsymbol{x}})\right]^2 , \end{aligned} \qquad (3\cdot 11)$$

[3] Proof of the relation (3·8) and a comment related to it are given in Appendix A.

with $1/\hat{r}^6 \stackrel{\rm def}{=} \lim_{\epsilon\to 0} 1/(\hat{r}^2+\epsilon^2)^3$. Hence, we have

$$\hat{R}^{\mu\nu}(\{\})\hat{R}_{\mu\nu}(\{\}) \neq 0 \ , \quad \hat{R}^{\rho\sigma\mu\nu}(\{\})\hat{R}_{\rho\sigma\mu\nu}(\{\}) \neq \frac{12a^2}{\hat{r}^6} \ . \tag{3·12}$$

In Eq. (3·11), the terms $(1/\hat{r}^3)\delta^{(3)}(\hat{\boldsymbol{x}})$ and $\left[\delta^{(3)}(\hat{\boldsymbol{x}})\right]^2$ are both ill-defined, and the symbol $\sim$ is to denote the "equality" of the left- and right-hand sides in a rough sense. Mathematical formalism capable of describing these singular quantities is needed.

It is worth mentioning here that the spherically symmetric non-scalar quantities $\hat{r}^3\hat{R}^{\mu\nu}(\{\};\epsilon)\hat{R}_{\mu\nu}(\{\};\epsilon)$ and $\hat{r}^3\hat{R}^{\rho\sigma\mu\nu}(\{\};\epsilon)\hat{R}_{\rho\sigma\mu\nu}(\{\};\epsilon)$ have the well-defined limits

$$\begin{aligned} \lim_{\epsilon\to 0}\hat{r}^3\hat{R}^{\mu\nu}(\{\};\epsilon)\hat{R}_{\mu\nu}(\{\};\epsilon) &= \frac{11}{4}\pi a^2\delta^{(3)}(\hat{\boldsymbol{x}}) \ , \\ \lim_{\epsilon\to 0}\hat{r}^3\hat{R}^{\rho\sigma\mu\nu}(\{\};\epsilon)\hat{R}_{\rho\sigma\mu\nu}(\{\};\epsilon) &= 12a^2\left[\frac{1}{\hat{r}^3}\right] + \frac{11}{2}\pi a^2\delta^{(3)}(\hat{\boldsymbol{x}}) \end{aligned} \tag{3·13}$$

with

$$\left[\frac{1}{\hat{r}^3}\right] \stackrel{\rm def}{=} \lim_{\epsilon\to 0}\frac{\hat{r}^3}{(\hat{r}^2+\epsilon^2)^3} \ . \tag{3·14}$$

4 Generalization

We now consider the problem in the coordinates (x^0, r, θ, ϕ) related to the Schwarzschild coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$ through the relation (1·4). In the coordinates (x^0, r, θ, ϕ), the metric has the form

$$\begin{aligned} ds^2 &= -A(dx^0)^2 - 2D dx^0 dr + (B+C)(dr)^2 \\ &\quad + Br^2\left[(d\theta)^2 + \sin^2\theta(d\phi)^2\right] \ , \end{aligned} \tag{4·1}$$

where we have defined

$$\begin{aligned} A &\stackrel{\rm def}{=} \Omega^2\left(1-\frac{a}{K}\right) \ , \quad B \stackrel{\rm def}{=} \frac{K^2}{\rho^2} \ , \\ C &\stackrel{\rm def}{=} \left(1-\frac{a}{K}\right)^{-1}(K')^2 - \frac{K^2}{\rho^2} - \left(1-\frac{a}{K}\right)(f')^2 \ , \\ D &\stackrel{\rm def}{=} \Omega\left(1-\frac{a}{K}\right)f' \end{aligned} \tag{4·2}$$

with[4] $\rho \stackrel{\rm def}{=} r$, $K' \stackrel{\rm def}{=} dK/dr$ and $f' \stackrel{\rm def}{=} df/dr$. The coordinates $t = x^0/c$ and r are time and space coordinates, respectively, only if

$$1-\frac{a}{K} > 0 \ , \quad \left(1-\frac{a}{K}\right)^{-1}(K')^2 - \left(1-\frac{a}{K}\right)(f')^2 > 0 \ . \tag{4·3}$$

[4]The manipulation to denote r with ρ is for the regularization procedure employed below.

In the Cartesian coordinate system $\{x^\mu ; \mu = 0,1,2,3\}$ with $x^1 = r\cos\phi\sin\theta$, $x^2 = r\sin\phi\sin\theta$, $x^3 = r\cos\theta$, the metric takes the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{4·4}$$

with $g_{\mu\nu}$ given by

$$g_{00} = -A\ , \quad g_{0\alpha} = -D\frac{x^\alpha}{r}\ , \quad g_{\alpha\beta} = B\delta^{\alpha\beta} + C\frac{x^\alpha x^\beta}{r^2}\ . \tag{4·5}$$

We can obtain expressions for the quantities $\widetilde{\boldsymbol{t}}_\mu{}^\nu$, $\widetilde{\boldsymbol{T}}_\mu{}^\nu$, $R(\{\})$, $R^{\mu\nu}(\{\})R_{\mu\nu}(\{\})$ and $R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\})$ in terms of A, B, C, D, which are enumerated in Appendix B.

By using Eqs. (4·2), (B·1) and (B·2), we can show that

$$\widetilde{\boldsymbol{t}}_\mu{}^\nu = 0\ , \tag{4·6}$$

if **Case 1**:

$$K(r) = \Lambda r\ , \tag{4·7}$$

or **Case 2**:

$$K(r) = \Lambda r + a\ , \quad f(r) = \Gamma r + \Xi \tag{4·8}$$

with Λ, Γ and Ξ being real constants. (For **Case 1**, $f(r)$ is arbitrary.)

In what follows, we restrict our consideration to the above two cases, **Case 1** and **Case 2** with positive[5] Λ. In order to treat the singularity at $r = 0$ in a distribution theoretical way, we employ the following regularization scheme:

R.1 First, replace K' with Λ in the expression of the function C.

R.2 Then, replace K' and K'' appearing in the process of calculation with $\Lambda\rho'$ and $\Lambda\rho''$, respectively.

R.3 Finally, replace the function ρ with $\sqrt{r^2 + \epsilon^2}$.

Then, we obtain the following results by using Eqs. (4·2), (B·1)$\sim$(B·4):

Case 1

[5]Note the following: (a) It is expected that a delta function expression for $\widetilde{\boldsymbol{T}}_\mu{}^\nu(x)$ is obtained when $\widetilde{\boldsymbol{t}}_\mu{}^\nu = 0$. (b) The variable r cannot be "*radial*" if $\Lambda \le 0$.

(1) The regularized energy-momentum density $\widetilde{\boldsymbol{T}}_\mu{}^\nu(x;\epsilon)$ is given by

$$\begin{aligned}
\kappa\widetilde{\boldsymbol{T}}_0{}^0(x;\epsilon) &= -|\Omega|\frac{a\epsilon^2}{r^2(r^2+\epsilon^2)^{3/2}} \ , \\
\kappa\widetilde{\boldsymbol{T}}_0{}^\alpha(x;\epsilon) &= 0 \ , \quad \kappa\widetilde{\boldsymbol{T}}_\alpha{}^0(x;\epsilon) = 0 \ , \\
\kappa\widetilde{\boldsymbol{T}}_\alpha{}^\beta(x;\epsilon) &= |\Omega|\left[\delta_\alpha{}^\beta\frac{3a\epsilon^2}{2(r^2+\epsilon^2)^{5/2}} - \frac{x^\alpha x^\beta}{r^2}\frac{a\epsilon^2}{(r^2+\epsilon^2)^{5/2}}\left(\frac{5}{2}+\frac{\epsilon^2}{r^2}\right)\right] \ .
\end{aligned} \tag{4·9}$$

This leads to

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu(x) \stackrel{\text{def}}{=} \lim_{\epsilon\to 0}\widetilde{\boldsymbol{T}}_\mu{}^\nu(x;\epsilon) = -Mc^2|\Omega|\delta_\mu{}^0\delta_0{}^\nu\delta^{(3)}(\boldsymbol{x}) \ , \tag{4·10}$$

which reduces to

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu(x) = -Mc^2\delta_\mu{}^0\delta_0{}^\nu\delta^{(3)}(\boldsymbol{x}) \tag{4·11}$$

when $|\Omega| = 1$. There is the relation

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu(x;\epsilon) = \left[\widetilde{\boldsymbol{T}}_\mu{}^\nu(x;\epsilon/\Lambda)\right]_c \ , \tag{4·12}$$

where we have defined

$$\left[\widetilde{\boldsymbol{T}}_\mu{}^\nu(x;\epsilon)\right]_c \stackrel{\text{def}}{=} \frac{\partial(\hat{x})}{\partial(x)}\frac{\partial\hat{x}^\lambda}{\partial x^\mu}\frac{\partial x^\nu}{\partial\hat{x}^\rho}\hat{\widetilde{\boldsymbol{T}}}_\lambda{}^\rho(\hat{x};\epsilon) \ . \tag{4·13}$$

Thus, we see that

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu(x) = \lim_{\epsilon\to 0}\left[\widetilde{\boldsymbol{T}}_\mu{}^\nu(x;\epsilon)\right]_c \ . \tag{4·14}$$

For

$$\left[\widetilde{\boldsymbol{T}}_\mu{}^\nu(x)\right]_c \stackrel{\text{def}}{=} \frac{\partial(\hat{x})}{\partial(x)}\frac{\partial\hat{x}^\lambda}{\partial x^\mu}\frac{\partial x^\nu}{\partial\hat{x}^\rho}\hat{\widetilde{\boldsymbol{T}}}_\lambda{}^\rho(\hat{x}) \ , \tag{4·15}$$

however, we have

$$\begin{aligned}
\left[\widetilde{\boldsymbol{T}}_0{}^0(x)\right]_c &= -Mc^2|\Omega|\delta^{(3)}(\boldsymbol{x}) \ , \quad \left[\widetilde{\boldsymbol{T}}_0{}^\alpha(x)\right]_c = 0 \ , \\
\left[\widetilde{\boldsymbol{T}}_\alpha{}^0(x)\right]_c &= -\frac{Mc^2|\Omega|}{\Omega}f'(r)\frac{x^\alpha}{r}\delta^{(3)}(\boldsymbol{x}) \ , \\
\left[\widetilde{\boldsymbol{T}}_\alpha{}^\beta(x)\right]_c &= 0 \ ,
\end{aligned} \tag{4·16}$$

in which the ill-defined quantity[6] $\frac{x^\alpha}{r}\delta^{(3)}(\boldsymbol{x})$ appears, and hence

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu(x) \neq \left[\widetilde{\boldsymbol{T}}_\mu{}^\nu(x)\right]_c \ . \tag{4·17}$$

[6]Note that $x^\alpha[(1/r)\delta^{(3)}(\boldsymbol{x})] \neq 0 = (1/r)[x^\alpha\delta^{(3)}(\boldsymbol{x})]$ and that $\left[\sum_{\alpha=1}^3 \frac{x^\alpha}{r}\frac{x^\alpha}{r}\right]\delta^{(3)}(\boldsymbol{x}) = \delta^{(3)}(\boldsymbol{x})$. See also the relation (A·16) in Appendix A.

From Eqs. (4·14) and (4·17), we see the following: The correct expression for the energy-momentum density $\widetilde{\boldsymbol{T}}_{\mu}^{\ \nu}(x)$ is obtainable by transforming *first* the regularized density $\hat{\widetilde{\boldsymbol{T}}}_{\mu}^{\ \nu}(\hat{x};\epsilon)$ and *then* taking the limit $\epsilon \to 0$. However, it cannot be obtained, if the order of making coordinate transformation and taking the limit is exchanged.

In Ref. [2], the expression $\widetilde{\boldsymbol{T}}_{\mu}^{\ \nu}(x) = -Mc^2\delta_{\mu}^{\ 0}\delta_{0}^{\ \nu}\delta^{(3)}(\boldsymbol{x})$ is given for Kerr-Schild coordinates. One might think that Eq. (1·3) can be easily obtained by transforming this to Schwarzschild coordinates. However, this is not the case; the situation here is similar to the one discussed above.

(2) For the regularized scalar quantities, we have

$$\begin{aligned} R(\{\};\epsilon) &= \Lambda^{-3}\left[-\frac{3a\epsilon^2}{(r^2+\epsilon^2)^{5/2}} + \frac{2a\epsilon^2}{r^2(r^2+\epsilon^2)^{3/2}}\right] = \hat{R}(\{\};\Lambda\epsilon)\ , \\ R^{\mu\nu}(\{\};\epsilon)R_{\mu\nu}(\{\};\epsilon) &= \Lambda^{-6}\left\{\frac{1}{2}\left[\frac{3a\epsilon^2}{(r^2+\epsilon^2)^{5/2}}\right]^2 + 2\left[\frac{a\epsilon^2}{r^2(r^2+\epsilon^2)^{3/2}}\right]^2\right\} \\ &= \hat{R}^{\mu\nu}(\{\};\Lambda\epsilon)\hat{R}_{\mu\nu}(\{\};\Lambda\epsilon)\ , \\ R^{\rho\sigma\mu\nu}(\{\};\epsilon)R_{\rho\sigma\mu\nu}(\{\};\epsilon) &= \Lambda^{-6}\left\{\frac{4a^2}{r^2+\epsilon^2}\left[\frac{2}{(r^2+\epsilon^2)^2} + \frac{1}{r^4}\right] - \frac{12a^2\epsilon^2}{(r^2+\epsilon^2)^4}\right. \\ &\qquad \left. + \frac{9a^2\epsilon^4}{(r^2+\epsilon^2)^5}\right\} \\ &= \hat{R}^{\rho\sigma\mu\nu}(\{\};\Lambda\epsilon)\hat{R}_{\rho\sigma\mu\nu}(\{\};\Lambda\epsilon)\ . \end{aligned} \tag{4·18}$$

Equation (4·18) implies that the regularized scalars in the coordinate system $\{x^{\mu};\mu = 0,1,2,3\}$ for **Case 1** essentially agree with the corresponding scalars in Schwarzschild coordinates.[7] This is not self-evident, because we have made the regularization in each coordinate system independently.[8]

Case 2

(1) The regularized energy-momentum density $\widetilde{\boldsymbol{T}}_{\mu}^{\ \nu}(x;\epsilon)$ is given by

$$\kappa\widetilde{\boldsymbol{T}}_{0}^{\ 0}(x;\epsilon) = |\Omega|\left[-\frac{6a\epsilon^2}{(r^2+\epsilon^2)^{5/2}} + \frac{a\epsilon^2}{r^2(r^2+\epsilon^2)^{3/2}}\right]\ ,$$

[7]Note that the product $\Lambda\epsilon$ in the hatted scalars in Eq. (4·18) plays essentially the same role as ϵ, because the limit $\epsilon \to 0$ is taken in the final stages.

[8]See also the paragraph at the end of this section.

$$
\begin{aligned}
\kappa \widetilde{\boldsymbol{T}}_0{}^{\alpha}(x;\epsilon) &= 0\ , \quad \kappa \widetilde{\boldsymbol{T}}_\alpha{}^{0}(x;\epsilon) = -\frac{|\Omega|}{\Omega}\Gamma\frac{x^\alpha}{r}\left[\frac{6a\epsilon^2}{(r^2+\epsilon^2)^{5/2}}\right]\ , \\
\kappa \widetilde{\boldsymbol{T}}_\alpha{}^{\beta}(x;\epsilon) &= |\Omega|\left[-\delta_\alpha{}^\beta\frac{3a\epsilon^2}{2(r^2+\epsilon^2)^{5/2}}+\frac{x^\alpha x^\beta}{r^2}\frac{a\epsilon^2}{(r^2+\epsilon^2)^{5/2}}\left(\frac{5}{2}+\frac{\epsilon^2}{r^2}\right)\right]\ ,
\end{aligned}
\tag{4·19}
$$

which yields a limit having the same form as Eq. (4·10), upon using Eqs. (3·7), (3·8) and the relation[9]

$$
\lim_{\epsilon\to 0}\left[\frac{x^\alpha}{r}\frac{\epsilon^2}{(r^2+\epsilon^2)^{5/2}}\right]=0\ . \tag{4·20}
$$

The quantity $\left[\widetilde{\boldsymbol{T}}_\mu{}^{\nu}(x;\epsilon)\right]_c$ defined in the same way as Eq. (4·13) is given by

$$
\begin{aligned}
\kappa\left[\widetilde{\boldsymbol{T}}_0{}^{0}(x;\epsilon)\right]_c &= -|\Lambda\Omega|\frac{a\epsilon^2}{r^2[(\Lambda r+a)^2+\epsilon^2]^{3/2}}\ , \\
\kappa\left[\widetilde{\boldsymbol{T}}_0{}^{\alpha}(x;\epsilon)\right]_c &= 0\ , \quad \kappa\left[\widetilde{\boldsymbol{T}}_\alpha{}^{0}(x;\epsilon)\right]_c = 0\ , \\
\kappa\left[\widetilde{\boldsymbol{T}}_\alpha{}^{\beta}(x;\epsilon)\right]_c &= |\Lambda\Omega|\left(\frac{\Lambda r+a}{r}\right)^2\left\{\frac{3a}{2}\frac{\epsilon^2}{[(\Lambda r+a)^2+\epsilon^2]^{5/2}}\delta_\alpha^\beta\right. \\
&\qquad \left.-\frac{a\epsilon^2}{[(\Lambda r+a)^2+\epsilon^2]^{3/2}}\frac{x^\alpha x^\beta}{r^2}\left[\frac{1}{(\Lambda r+a)^2}+\frac{3}{2}\frac{1}{(\Lambda r+a)^2+\epsilon^2}\right]\right\}\ ,
\end{aligned}
\tag{4·21}
$$

from which the limit

$$
\lim_{\epsilon\to 0}\left[\widetilde{\boldsymbol{T}}_\mu{}^{\nu}(x;\epsilon)\right]_c = 0 \neq \widetilde{\boldsymbol{T}}_\mu{}^{\nu}(x) \tag{4·22}
$$

is obtained. Also, $\left[\widetilde{\boldsymbol{T}}_\mu{}^{\nu}(x)\right]_c$ defined in the same way as for **Case 1** has the expression

$$
\begin{aligned}
\left[\widetilde{\boldsymbol{T}}_0{}^{0}(x)\right]_c &= -Mc^2|\Lambda\Omega|\left(\frac{\Lambda r+a}{r}\right)^2\delta^{(3)}(\Lambda\boldsymbol{x}+\boldsymbol{a})\ , \\
\left[\widetilde{\boldsymbol{T}}_0{}^{\alpha}(x)\right]_c &= 0\ , \quad \left[\widetilde{\boldsymbol{T}}_\alpha{}^{\beta}(x)\right]_c = 0\ , \\
\left[\widetilde{\boldsymbol{T}}_\alpha{}^{0}(x)\right]_c &= -Mc^2\frac{|\Lambda\Omega|}{\Omega}\Gamma\left(\frac{\Lambda r+a}{r}\right)^2\frac{x^\alpha}{r}\delta^{(3)}(\Lambda\boldsymbol{x}+\boldsymbol{a})
\end{aligned}
\tag{4·23}
$$

with $\boldsymbol{a} \stackrel{\mathrm{def}}{=} a\boldsymbol{x}/r$, and we have a relation having the same form as Eq. (4·17). This and Eq. (4·22) imply that the correct expression for $\widetilde{\boldsymbol{T}}_\mu{}^{\nu}(x)$ is *not* obtainable for

[9] Proof of the relation (4·20) and a comment related to it are given in Appendix A.

Case 2 by applying the coordinate transformation and the limiting procedure to the regularized density $\hat{\tilde{\boldsymbol{T}}}_\mu{}^\nu(\hat{x};\epsilon)$.

(2) The regularized scalar quantities are given by

$$\begin{aligned}
R(\{\};\epsilon) &= \frac{r^2+\epsilon^2}{\Lambda(\Lambda\sqrt{r^2+\epsilon^2}+a)^2}\left[\frac{9a\epsilon^2}{(r^2+\epsilon^2)^{5/2}}-\frac{2a\epsilon^2}{r^2(r^2+\epsilon^2)^{3/2}}\right] , \\
R^{\mu\nu}(\{\};\epsilon)R_{\mu\nu}(\{\};\epsilon) &= \frac{(r^2+\epsilon^2)^2}{\Lambda^2(\Lambda\sqrt{r^2+\epsilon^2}+a)^4}\left\{\frac{5}{2}\left[\frac{3a\epsilon^2}{(r^2+\epsilon^2)^{5/2}}\right]^2\right. \\
&\left. +2\left[-\frac{3a\epsilon^2}{(r^2+\epsilon^2)^{5/2}}+\frac{a\epsilon^2}{r^2(r^2+\epsilon^2)^{3/2}}\right]^2\right\} , \\
R^{\rho\sigma\mu\nu}(\{\};\epsilon)R_{\rho\sigma\mu\nu}(\{\};\epsilon) &= \frac{12a^2}{\Lambda^2(\Lambda\sqrt{r^2+\epsilon^2}+a)^6}\left[\Lambda+\frac{a\epsilon^2}{(r^2+\epsilon^2)^{3/2}}\right]^2 \\
&-\frac{4a^2}{\Lambda^2(\Lambda\sqrt{r^2+\epsilon^2}+a)^5}\left[\Lambda+\frac{a\epsilon^2}{(r^2+\epsilon^2)^{3/2}}\right] \\
&\times\left[\frac{2\epsilon^2}{r^2\sqrt{r^2+\epsilon^2}}+\frac{9\epsilon^2}{(r^2+\epsilon^2)^{3/2}}\right] \\
&+\frac{a^2}{\Lambda^2(\Lambda\sqrt{r^2+\epsilon^2}+a)^4}\left[\frac{4\epsilon^4}{r^4(r^2+\epsilon^2)}+\frac{81\epsilon^4}{(r^2+\epsilon^2)^3}\right] ,
\end{aligned} \tag{4·24}$$

from which the following is known:

$$\begin{aligned}
R(\{\}) &\stackrel{\text{def}}{=} \lim_{\epsilon\to 0} R(\{\};\epsilon) = 0 , \\
R^{\mu\nu}(\{\};\epsilon)R_{\mu\nu}(\{\};\epsilon) &\sim \frac{2a^2}{\Lambda^2(\Lambda r+a)^4}\frac{\epsilon^4}{r^4(r^2+\epsilon^2)} \neq 0 , \\
R^{\rho\sigma\mu\nu}(\{\};\epsilon)R_{\rho\sigma\mu\nu}(\{\};\epsilon) &\sim \frac{12a^2}{(\Lambda r+a)^6}+\frac{4a^2}{\Lambda^2(\Lambda r+a)^4}\frac{\epsilon^4}{r^4(r^2+\epsilon^2)} \\
&\neq \frac{12a^2}{(\Lambda r+a)^6} .
\end{aligned} \tag{4·25}$$

Also, we can show that

$$\begin{aligned}
\lim_{\epsilon\to 0} r^\omega R^{\mu\nu}(\{\};\epsilon)R_{\mu\nu}(\{\};\epsilon) &= 0 , \\
\lim_{\epsilon\to 0} r^\omega R^{\rho\sigma\mu\nu}(\{\};\epsilon)R_{\rho\sigma\mu\nu}(\{\};\epsilon) &= 0 , \quad \text{for } \omega > 1 .
\end{aligned} \tag{4·26}$$

As is known from Eqs. (3·9) and (4·24), the regularized scalars in the coordinate system $\{x^\mu;\mu=0,1,2,3\}$ for **Case 2** are not simply related to the corresponding

scalars in Schwarzschild coordinates. However, this is *not* a contradiction, because the regularization has been made in each coordinate system independently.

5 Summary and comments

In the above, we have examined the Schwarzschild space-time from a distribution theoretical point of view. Expressions of the energy-momentum densities $\widetilde{\boldsymbol{T}}_{\mu}^{\ \nu}$ and $\widetilde{\boldsymbol{t}}_{\mu}^{\ \nu}$ and of the scalars $R(\{\}), R^{\mu\nu}(\{\})R_{\mu\nu}(\{\})$ and $R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\})$ have been obtained for a class of coordinate systems including those of the Schwarzschild and of Kerr-Schild types as special cases.

The results can be summarized as follows:

(1) For Schwarzschild coordinates, employing the regularization scheme prescribed by the replacement $a/\hat{r}$ with $a/\sqrt{\hat{r}^2+\epsilon^2}$ in the expression (3·3) of the components of the metric, we have shown the following:

(1.A) The energy-momentum density $\hat{\widetilde{\boldsymbol{T}}}_{\mu}^{\ \nu}(\hat{x})$ of the gravitational source has the expression (1·3).

(1.B) The scalar $\hat{R}(\{\})$ has the definite delta function expression (3·10), while $\hat{R}^{\mu\nu}(\{\})\hat{R}_{\mu\nu}(\{\})$ and $\hat{R}^{\rho\sigma\mu\nu}(\{\})\hat{R}_{\rho\sigma\mu\nu}(\{\})$ have the expression (3·11), which include terms like $\left[\delta^{(3)}(\boldsymbol{x})\right]^2$ and $\delta^{(3)}(\boldsymbol{x})/r^3$. Also, $\hat{r}^3\hat{R}^{\mu\nu}(\{\};\epsilon)\hat{R}_{\mu\nu}(\{\};\epsilon)$ and $\hat{r}^3\hat{R}^{\rho\sigma\mu\nu}(\{\};\epsilon)\hat{R}_{\rho\sigma\mu\nu}(\{\};\epsilon)$ have the definite limits (3·13). The second relation in Eq. (3·12) shows that the well-known expression $\hat{R}^{\rho\sigma\mu\nu}(\{\})\hat{R}_{\rho\sigma\mu\nu}(\{\}) = 12a^2/\hat{r}^6$ is not correct, as long as we define $1/\hat{r}^6 \stackrel{\text{def}}{=} \lim_{\epsilon\to 0} 1/(\hat{r}^2+\epsilon^2)^3$.

(2) For the coordinates (x^0, r, θ, ϕ) related to the Schwarzschild coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$ through (1.4), we have shown the following:

(2.A) If **Case 1**: $K(r) = \Lambda r$, or **Case 2**: $K(r) = \Lambda r + a$, $f(r) = \Gamma r + \Xi$ with Λ, Γ and Ξ being real constants, the gravitational energy-momentum density vanishes, $\widetilde{\boldsymbol{t}}_{\mu}^{\ \nu} = 0$.

(2.B) Restricting ourselves to cases with positive Λ and employing the regularization scheme prescribed by **R.1**, **R.2** and **R.3**, we have shown the following:
For both **Case 1** and **Case 2**, the energy-momentum density $\widetilde{\boldsymbol{T}}_{\mu}^{\ \nu}(x)$ has the

delta function expression (4·10), which reduces to $\widetilde{\boldsymbol{T}}_{\mu}{}^{\nu}(x) = -Mc^2\delta_\mu{}^0\delta_0{}^\nu\delta^{(3)}(\boldsymbol{x})$, when $|\Omega| = 1$.

Case 1

(2.1.a) The correct expression for $\widetilde{\boldsymbol{T}}_{\mu}{}^{\nu}(x)$ can be obtained by transforming *first* the regularized density $\hat{\widetilde{\boldsymbol{T}}}_{\mu}{}^{\nu}(\hat{x};\epsilon)$ and *then* taking the limit $\epsilon \to 0$. However, it cannot be obtained if the order of the coordinate transformation and the limiting procedure is exchanged.

(2.1.b) The regularized scalars $R(\{\};\epsilon)$, $R^{\mu\nu}(\{\};\epsilon)R_{\mu\nu}(\{\};\epsilon)$ and $R^{\rho\sigma\mu\nu}(\{\};\epsilon)$ $\times R_{\rho\sigma\mu\nu}(\{\};\epsilon)$ are essentially equal to the corresponding scalars in Schwarzschild coordinates.

Case 2

(2.2.a) The correct expression for $\widetilde{\boldsymbol{T}}_{\mu}{}^{\nu}(x)$ cannot be obtained by considering the transformation of the regularized or limiting densities in Schwarzschild coordinates.

(2.2.b) The regularized scalars $R(\{\};\epsilon)$, $R^{\mu\nu}(\{\};\epsilon)R_{\mu\nu}(\{\};\epsilon)$ and $R^{\rho\sigma\mu\nu}(\{\};\epsilon)$ $\times R_{\rho\sigma\mu\nu}(\{\};\epsilon)$ are given by Eq. (4·24), from which Eqs. (4·25) and (4·26) follow.

The following is worth mentioning:

[A] When the condition $|\Omega| = 1$ is satisfied, the gravity appears to be interpreted as produced by a point-like particle of mass M located at $\boldsymbol{x} = \boldsymbol{0}$ for both **Case 1** and **Case 2**, so far as energy-momentum densities are concerned. Also for the orbit of a test point-like particle moving in the region $r > a$, this interpretation works well for **Case 1** if the additional condition[10]

$$|f'|\left(1 - \frac{a}{r}\right) < 1\ , \tag{5·1}$$

following from Eq. (4·3), is satisfied. But, for **Case 2**, a test point-like particle is expected to move as if the source were located at $r = -a/\Lambda$. The fact that energy-momentum density $\widetilde{\boldsymbol{T}}_{\mu}{}^{\nu}(x)$ takes the form (4·11) does not necessarily imply that the gravity is produced by a mass point located at $\boldsymbol{x} = \boldsymbol{0}$.

[10]Note that the condition (5·1) is satisfied for Schwarzschild and Kerr-Schild coordinates.

[B] The situation is simple and natural in **Case 1**, as in Schwarzschild coordinates. This is reasonable, because the Jacobian of the transformation (1·4) for this case is given by

$$\frac{\partial(x')}{\partial(x)} = \Omega\Lambda^3 \ , \tag{5·2}$$

which implies that the transformation is everywhere regular.

[C] The volume element $\sqrt{-g}dx^0dx^1dx^2dx^3$ is given by

$$\sqrt{-g}dx^0dx^1dx^2dx^3 = |\Omega\Lambda^3|dx^0dx^1dx^2dx^3 \ , \tag{5·3}$$

for **Case 1**, which includes Schwarzschild coordinates and Kerr-Schild coordinates as special cases. This case describes volume-preserving coordinates, which are preferred in the sense mentioned in Ref. [5].

[D] For **Case 2**, the situation is rather curious, as is known from the statement **[A]**. The following is worth noting in this connection: The Jacobian of the coordinate transformation (1·4) for this case is

$$\frac{\partial(x')}{\partial(x)} = \Omega\Lambda\frac{(\Lambda r + a)^2}{r^2} \ , \tag{5·4}$$

which means that the coordinate transformation is singular at $r = 0$ and at $\Lambda r + a = 0$. Thus, the coordinate systems of **Case 2** and of Schwarzschild are not equivalent, and the corresponding metrics describe different physical situations.[6)]

[E] The singularity at $\boldsymbol{x} = \boldsymbol{0}$ in **Case 2** is weaker than in the case of the Schwarzschild coordinate system and in **Case 1**, as is seen from Eqs. (3·10), (3·11), (3·13), (4·18), (4·25) and (4·26). This is expected from the regularity and singularity of the Jacobians (5·2) and (5·4).

[F] We must be careful in considering the coordinate transformation of distributional quantities, as is known from Eqs. (3·9), (3·10), (3·11), (4·24), (4·25) and the statements **(2.1.a)** and **(2.2.a)**.

[G] The regularization schemes employed in §§ 3 and 4 are crucial in giving our results. We have regularized the "dynamical part" only. If we had regularized the "kinematical factors" $1/\hat{r}$ and $1/\hat{r}^2$ in Eq. (3·4) as $1/\sqrt{\hat{r}^2 + \epsilon^2}$ and $1/(\hat{r}^2 + \epsilon^2)$, respectively, in

addition to regularizing the "dynamical function" $h = a/\hat{r}$, for example, we would not have obtained Eq. (1·3). Also, if we had not followed the scheme prescribed by **R.1**, **R.2** and **R.3** in §4, we might have been led to unwelcome results. However, as for this scheme, we cannot simply say that we have regularized "dynamical parts" only. At present, we cannot give a lucid interpretation to the scheme employed in §4.

[H] Sometimes, a singularity is considered as a point existing outside of space-time and is thus removed from consideration. However, this way of handling such entities is not appropriate, because then the Schwarzschild gravity described by Eq. (1·1), for example, has to be regarded as produced by zero energy-momentum.[1] A singularity exists in the space-time manifold, although it is not a "place".[7] We believe it constructive to examine space-time singularities by extending the analysis on manifolds in a distribution theoretical way, although we are not equipped with satisfactory mathematical machinery as yet. The expressions for scalar concomitants of the curvature tensor imply that mathematical formalism capable of incorporating objects like $\left[\delta^{(3)}(\boldsymbol{x})\right]^2$ and $\delta^{(3)}(\boldsymbol{x})/r^3$ is needed for a precise description of singularities.

[I] We could not construct well-defined expressions for the quadratic scalars $\hat{R}^{\mu\nu}(\{\})\times\hat{R}_{\mu\nu}(\{\})$ and $\hat{R}^{\rho\sigma\mu\nu}(\{\})\hat{R}_{\rho\sigma\mu\nu}(\{\})$. Otherwise, however, our prescription leads to reasonable results, such as Eqs. (1·3) and (3·10). Thus, the claim, "the well-known expression $\hat{R}^{\rho\sigma\mu\nu}(\{\})\hat{R}_{\rho\sigma\mu\nu}(\{\}) = 12a^2/\hat{r}^6$ is not correct $\cdots$" in the statement (1.B), is well-grounded.

[J] Both **Case 1** and **Case 2** do not satisfy the harmonic condition, and **Case 1** with $\Lambda = 1 = |\Omega|$ is suited for the description of the Schwarzschild gravity and its source. This is opposed to the argument in favor of the harmonic condition as a physical supplement to Einstein's theory of gravitation.[8, 9, 10, 11] For Lanczos coordinates, which satisfy the harmonic condition, the energy-momentum density of the source does not take the simple form Eq. (4·10). Harmonic coordinates are suited to discuss the gravitational wave, but they are not suited for the description of the Schwarzschild space-time.

Appendix A

— ***Proofs of relations (3·8) and (4·20)*** —

First, we prove the relation (3·8). Let us define[11]

$$F^{\alpha\beta}(\boldsymbol{x};\epsilon) \stackrel{\rm def}{=} \frac{x^\alpha x^\beta}{r^2}\left[\frac{3}{2}\frac{\epsilon^2}{(r^2+\epsilon^2)^{5/2}} + \frac{\epsilon^2}{r^2(r^2+\epsilon^2)^{3/2}}\right] , \qquad (\mathrm{A\cdot 1})$$

and let $\Phi(\boldsymbol{x})$ be an arbitrary function of class C^∞ with compact support:

$$r^2\Phi^{\alpha\beta}(r) \stackrel{\rm def}{=} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} x^\alpha x^\beta \Phi(\boldsymbol{x}) d\phi = 0 \ , \ \text{for } r \geq R \qquad (\mathrm{A\cdot 2})$$

with R being a positive constant. Then, we have

$$I^{\alpha\beta}(\epsilon) \stackrel{\rm def}{=} \int F^{\alpha\beta}(\boldsymbol{x};\epsilon)\Phi(\boldsymbol{x})d^3\boldsymbol{x} = \int_0^{\frac{R}{\epsilon}} \Phi^{\alpha\beta}(\epsilon x)\left[\frac{5}{2}\frac{1}{(x^2+1)^{3/2}} - \frac{3}{2}\frac{1}{(x^2+1)^{5/2}}\right] dx \ , \qquad (\mathrm{A\cdot 3})$$

which is calculated to give

$$\begin{aligned} I^{\alpha\beta}(\epsilon) &= \sum_{l=0}^{n-1} \frac{\Phi^{\alpha\beta(l)}(0)}{l!}\epsilon^l \left[\frac{5}{4}B_{\frac{R^2}{R^2+\epsilon^2}}\left(\frac{1+l}{2}, 1-\frac{l}{2}\right) - \frac{3}{4}B_{\frac{R^2}{R^2+\epsilon^2}}\left(\frac{1+l}{2}, 2-\frac{l}{2}\right)\right] \\ &\quad + \frac{\epsilon^n}{n!}\int_0^{\frac{R}{\epsilon}} \Phi^{\alpha\beta(n)}(\xi)x^n \left[\frac{5}{2}\frac{1}{(x^2+1)^{3/2}} - \frac{3}{2}\frac{1}{(x^2+1)^{5/2}}\right] dx \ , \qquad (\mathrm{A\cdot 4}) \end{aligned}$$

where we have expressed the function $\Phi^{\alpha\beta}(\epsilon x)$ as

$$\begin{aligned} \Phi^{\alpha\beta}(\epsilon x) &= \sum_{l=0}^{n-1} \frac{\Phi^{\alpha\beta(l)}(0)}{l!}(\epsilon x)^l + \frac{1}{n!}(\epsilon x)^n \Phi^{\alpha\beta(n)}(\xi) \ , \\ &\quad \xi \stackrel{\rm def}{=} \theta\epsilon x \ , \quad 1 > \theta > 0 \ , \quad n \geq 1 \qquad (\mathrm{A\cdot 5}) \end{aligned}$$

with $\Phi^{\alpha\beta(l)} \stackrel{\rm def}{=} d^l\Phi^{\alpha\beta}/dr^l$. Also, $B_z(p,q)$ represents the incomplete beta function of the first kind:

$$B_z(p,q) \stackrel{\rm def}{=} \int_0^z t^{p-1}(1-t)^{q-1}dt \ , \quad 1 > \mathrm{Re}\, z > 0 \ . \qquad (\mathrm{A\cdot 6})$$

Equation (A·4) gives

$$\lim_{\epsilon\to 0} I^{\alpha\beta}(\epsilon) = 2\pi\delta^{\alpha\beta}\Phi(\boldsymbol{x}=\boldsymbol{0}) \ , \qquad (\mathrm{A\cdot 7})$$

[11] In this Appendix, the symbol "ˆ" is omitted for simplicity.

where use is made of the relation

$$\Phi^{\alpha\beta}(0) = \frac{4}{3}\pi\delta^{\alpha\beta}\Phi(\boldsymbol{x} = \mathbf{0}) \ . \tag{A·8}$$

Thus, we see that $\lim_{\epsilon\to 0} F^{\alpha\beta}(\boldsymbol{x};\epsilon) = 2\pi\delta^{\alpha\beta}\delta^{(3)}(\boldsymbol{x})$. This is equivalent to Eq. (3·8).

Next, we prove Eq. (4· 20). Let us define

$$G^{\alpha}(\boldsymbol{x};\epsilon) \stackrel{\text{def}}{=} \frac{x^\alpha}{r}\frac{\epsilon^2}{(r^2+\epsilon^2)^{5/2}} \ , \tag{A·9}$$

and let $\Psi(\boldsymbol{x})$ be an arbitrary function of class C^∞ with compact support:

$$r\Psi^{\alpha}(r) \stackrel{\text{def}}{=} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} x^\alpha \Psi(\boldsymbol{x}) d\phi = 0 \ , \quad \text{for} \ \ r \geq R > 0 \ . \tag{A·10}$$

Then, we have

$$J^{\alpha}(\epsilon) \stackrel{\text{def}}{=} \int G^{\alpha}(\boldsymbol{x};\epsilon)\Psi(\boldsymbol{x}) d^3\boldsymbol{x} = \int_0^{\frac{R}{\epsilon}} \Psi^{\alpha}(\epsilon x)\frac{x^2}{(x^2+1)^{5/2}} dx \ , \tag{A·11}$$

from which the expression

$$\begin{aligned} J^{\alpha}(\epsilon) \ &= \ \frac{1}{2}\sum_{l=0}^{n-1}\frac{\Psi^{\alpha(l)}(0)}{l!}\epsilon^l B_{\frac{R^2}{R^2+\epsilon^2}}\left(\frac{3+l}{2}, 1-\frac{l}{2}\right) \\ &\quad + \frac{\epsilon^n}{n!}\int_0^{\frac{R}{\epsilon}} \Psi^{\alpha(n)}(\xi)\frac{x^{n+2}}{(x^2+1)^{5/2}} dx \ , \quad \theta R \geq \xi \geq 0 \ , \end{aligned} \tag{A·12}$$

follows, where we have expressed the function $\Psi^{\alpha}(\epsilon x)$ as

$$\begin{aligned} \Psi^{\alpha}(\epsilon x) \ &= \ \sum_{l=0}^{n-1}\frac{\Psi^{\alpha(l)}(0)}{l!}(\epsilon x)^l + \frac{1}{n!}(\epsilon x)^n \Psi^{\alpha(n)}(\xi) \ , \\ &\quad \xi \stackrel{\text{def}}{=} \theta\epsilon x \ , \quad 1 > \theta > 0 \ , \quad n \geq 1 \end{aligned} \tag{A·13}$$

with $\Psi^{\alpha(l)} \stackrel{\text{def}}{=} d^l\Psi^\alpha/dr^l$. Equation (A·12) leads to $\lim_{\epsilon\to 0} J^\alpha(\epsilon) = \Psi^\alpha(0)/3 = 0$, which gives Eq. (4·20).

We note that the definite relation

$$\lim_{\epsilon\to 0}\sum_{\alpha,\,\beta=1}^{3}\frac{x^\alpha x^\beta}{r^2}F^{\alpha\beta}(\boldsymbol{x};\epsilon) = 6\pi\delta^{(3)}(\boldsymbol{x}) \ , \tag{A·14}$$

exists, while, the limit

$$\sum_{\alpha,\,\beta=1}^{3}\frac{x^\alpha x^\beta}{r^2}\lim_{\epsilon\to 0}F^{\alpha\beta}(\boldsymbol{x};\epsilon) \ , \tag{A·15}$$

is not well-defined, as is known from the footnote 6 on page 9. Also, we have

$$\lim_{\epsilon\to 0}\sum_{\alpha=1}^{3}\frac{x^\alpha}{r}G^\alpha(\boldsymbol{x};\epsilon) \;=\; \frac{4}{3}\pi\delta^{(3)}(\boldsymbol{x}) \neq 0 = \sum_{\alpha=1}^{3}\frac{x^\alpha}{r}\lim_{\epsilon\to 0}G^\alpha(\boldsymbol{x};\epsilon)\ . \tag{A·16}$$

We must be careful in treating $F^{\alpha\beta}(\boldsymbol{x};\epsilon)$ and $G^\alpha(\boldsymbol{x};\epsilon)$.

Appendix B

— ***Expressions for the quantities*** $\widetilde{\boldsymbol{t}}_\mu{}^\nu$, $\widetilde{\boldsymbol{T}}_\mu{}^\nu$, $R(\{\})$, $R^{\mu\nu}(\{\})R_{\mu\nu}(\{\})$ ***and*** $R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\})$ ***in terms of*** A, B, C ***and*** D —

The energy-momentum density $\widetilde{\boldsymbol{t}}_\mu{}^\nu$ is expressed as

$$\begin{aligned}
2\kappa\widetilde{\boldsymbol{t}}_0{}^\mu &= \mathbb{G}\delta_0{}^\mu\ ,\quad 2\kappa\widetilde{\boldsymbol{t}}_\alpha{}^0 = \frac{BD}{\sqrt{\Delta}}\left[\frac{2B'D'}{BD}+\left(\frac{B'}{B}\right)^2+\frac{1}{r}\frac{\Delta'}{\Delta}\right]\frac{x^\alpha}{r}\ ,\\
2\kappa\widetilde{\boldsymbol{t}}_\alpha{}^\beta &= \mathbb{G}\delta_\alpha{}^\beta + \frac{AB}{\sqrt{\Delta}}\left\{\frac{1}{2r}\frac{AC+D^2}{AB}\frac{\Delta'}{\Delta}\left(\delta_\alpha{}^\beta+\frac{x^\alpha x^\beta}{r^2}\right)\right.\\
&\qquad\left. -\left[2\frac{A'B'}{AB}+\left(\frac{B'}{B}\right)^2\right]\frac{x^\alpha x^\beta}{r^2}\right\}\ ,
\end{aligned} \tag{B·1}$$

where $\mathbb{G}$ is given by

$$\mathbb{G} = \frac{AB}{\sqrt{\Delta}}\left[\frac{A'B'}{AB}+\frac{1}{2}\left(\frac{B'}{B}\right)^2-\frac{1}{r}\frac{AC+D^2}{AB}\frac{\Delta'}{\Delta}\right]\ , \tag{B·2}$$

and $\Delta \overset{\text{def}}{=} A(B+C)+D^2$. Also, for $\widetilde{\boldsymbol{T}}_\mu{}^\nu$, we have

$$\begin{aligned}
\kappa\widetilde{\boldsymbol{T}}_0{}^0 &= \frac{AB}{\sqrt{\Delta}}\left[\frac{1}{r}\left(\frac{A'}{A}+3\frac{B'}{B}-\frac{\Delta'}{\Delta}\right)-\frac{1}{r^2}\frac{AC+D^2}{AB}+\frac{B''}{B}\right.\\
&\qquad\left.-\frac{1}{4}\left(\frac{B'}{B}\right)^2+\frac{1}{2}\frac{A'B'}{AB}-\frac{1}{2}\frac{B'\Delta'}{B\Delta}\right],\\
\kappa\widetilde{\boldsymbol{T}}_0{}^\alpha &= 0,\quad \kappa\widetilde{\boldsymbol{T}}_\alpha{}^0 = \frac{x^\alpha}{r}\frac{BD}{\sqrt{\Delta}}\left[\frac{1}{r}\left(2\frac{B'}{B}-\frac{\Delta'}{\Delta}\right)+\frac{B''}{B}-\frac{1}{2}\left(\frac{B'}{B}\right)^2-\frac{1}{2}\frac{B'\Delta'}{B\Delta}\right],\\
\kappa\widetilde{\boldsymbol{T}}_\alpha{}^\beta &= \delta_\alpha{}^\beta\frac{AB}{\sqrt{\Delta}}\left[\frac{1}{r}\left(\frac{A'}{A}+\frac{B'}{B}-\frac{1}{2}\frac{\Delta'}{\Delta}\right)+\frac{1}{2}\frac{A''}{A}+\frac{1}{2}\frac{B''}{B}-\frac{1}{4}\left(\frac{B'}{B}\right)^2\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{A'B'}{AB}-\frac{1}{4}\left(\frac{A'}{A}+\frac{B'}{B}\right)\frac{\Delta'}{\Delta}\Bigg] \\
&+\frac{x^\alpha x^\beta}{r^2}\frac{AB}{\sqrt{\Delta}}\Bigg[\frac{1}{2r}\frac{\Delta'}{\Delta}-\frac{1}{r^2}\frac{AC+D^2}{AB}-\frac{1}{2}\frac{A''}{A}-\frac{1}{2}\frac{B''}{B} \\
&\qquad +\frac{1}{2}\left(\frac{B'}{B}\right)^2+\frac{1}{4}\left(\frac{A'}{A}+\frac{B'}{B}\right)\frac{\Delta'}{\Delta}\Bigg]. \qquad \text{(B·3)}
\end{aligned}
$$

For $R(\{\}), R^{\mu\nu}(\{\})R_{\mu\nu}(\{\})$ and $R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\})$, we have

$$
\begin{aligned}
R(\{\}) &= \frac{A}{\Delta}\Bigg[\frac{2}{r}\left(-2\frac{A'}{A}-3\frac{B'}{B}+\frac{\Delta'}{\Delta}\right)+\frac{2}{r^2}\frac{AC+D^2}{AB}-\frac{A''}{A}-2\frac{B''}{B} \\
&\qquad +\frac{1}{2}\left(\frac{B'}{B}\right)^2-2\frac{A'B'}{AB}+\left(\frac{1}{2}\frac{A'}{A}+\frac{B'}{B}\right)\frac{\Delta'}{\Delta}\Bigg], \\
R^{\mu\nu}(\{\})R_{\mu\nu}(\{\}) &= \frac{A^2}{\Delta^2}\left(\frac{1}{2}\frac{A''}{A}-\frac{1}{4}\frac{A'\Delta'}{A\Delta}+\frac{1}{2}\frac{A'B'}{AB}+\frac{1}{r}\frac{A'}{A}\right)^2 \\
&\quad +2\frac{A^2}{\Delta^2}\Bigg[\frac{1}{r}\left(\frac{1}{2}\frac{\Delta'}{\Delta}-\frac{A'}{A}-2\frac{B'}{B}\right)+\frac{1}{r^2}\frac{AC+D^2}{AB}-\frac{1}{2}\frac{A'B'}{AB} \\
&\qquad -\frac{1}{2}\frac{B''}{B}+\frac{1}{4}\frac{B'\Delta'}{B\Delta}\Bigg]^2 \\
&\quad +\frac{A^2}{\Delta^2}\Bigg[\frac{1}{2}\frac{A''}{A}-\frac{1}{4}\frac{A'\Delta'}{A\Delta}+\frac{1}{2}\frac{A'B'}{AB}+\frac{B''}{B}-\frac{1}{2}\left(\frac{B'}{B}\right)^2 \\
&\qquad -\frac{1}{2}\frac{B'\Delta'}{B\Delta}+\frac{1}{r}\left(\frac{A'}{A}-\frac{\Delta'}{\Delta}+2\frac{B'}{B}\right)\Bigg]^2, \\
R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\}) &= \frac{A^2}{\Delta^2}\left(\frac{A''}{A}-\frac{1}{2}\frac{A'\Delta'}{A\Delta}\right)^2+2\frac{A^2}{\Delta^2}\left(\frac{1}{r}\frac{A'}{A}+\frac{1}{2}\frac{A'B'}{AB}\right)^2 \\
&\quad +4\frac{A^2}{\Delta^2}\left[\frac{1}{r}\frac{B'}{B}-\frac{1}{r^2}\frac{AC+D^2}{AB}+\frac{1}{4}\left(\frac{B'}{B}\right)^2\right]^2 \\
&\quad +2\frac{A^2}{\Delta^2}\Bigg[\frac{1}{r}\left(\frac{A'}{A}+2\frac{B'}{B}-\frac{\Delta'}{\Delta}\right)+\frac{1}{2}\frac{A'B'}{AB}+\frac{B''}{B} \\
&\qquad -\frac{1}{2}\left(\frac{B'}{B}\right)^2-\frac{1}{2}\frac{B'\Delta'}{B\Delta}\Bigg]^2. \qquad \text{(B·4)}
\end{aligned}
$$

In the above, we have defined $A' \stackrel{\text{def}}{=} dA/dr, A'' \stackrel{\text{def}}{=} d^2A/dr^2$, etc.

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